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THE WHITEHEAD TORSION OF THE TOTAL SPACE OF A FIBER BUNDLE

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LET $F \to E \to B$ be a fiber bundle and let $A \subset B$ be a deformation retract of B. The covering homotopy property then shows that $E_A = p^{-1}(A)$ is a deformation retract of E. It is the object of this paper to begin the study of the natural question: "How are the Whitehead torsions $\tau(B, A) \in \mathrm{Wh}(\pi_1(B))$ and $\tau(E, E_A) \in \mathrm{Wh}(\pi_1(E))$ related?" (For an exposition of the basic properties of Whitehead torsion see [4], [7], [8] and [9].) In Section 3, we prove

THEOREM A. *Let $F \to E \to B$ be a fiber bundle with connected fiber F such that $H_*(F; Z)$ is torsion free. Then the action of $\pi_1(B)$ on $H_i(F; Z)$ determines an endomorphism σ_{i*}: $\mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(B))$ and*

$$p_*\tau(E, E_A) = \sum(-1)^i \sigma_{i*}\tau(B, A).$$

p_* denotes the map $p_*: \mathrm{Wh}(\pi_1(E)) \to \mathrm{Wh}(\pi_1(B))$ induced by $p_\#: \pi_1(E) \to \pi_1(B)$.

In some cases it is possible to describe the endomorphisms σ_{i*} explicitly. For example, if $\pi_1(B)$ acts trivially on $H_i(F; Z)$, then σ_{i*} is just multiplication by the rank of $H_i(F; Z)$. Hence

COROLLARY B. *If $F \to E \to B$ is an orientable fiber bundle with connected fiber F such that $H_*(F; Z)$ is torsion free, then*

$$p_*\tau(E, E_A) = \chi(F)\tau(B, A).$$

Now let π be a group and $\sigma: \pi \to \{\pm 1\}$ be a homomorphism. Then σ induces an involution of $Z(\pi)$ by sending $\lambda = \sum_{\alpha \in \pi} n_\alpha \alpha$ to $\lambda^* = \sum_{\alpha \in \pi} n_\alpha \sigma(\alpha)\alpha$ where $n_\alpha \in Z$ and therefore σ induces an involution of $\mathrm{Wh}(\pi)$ which we also denote by *.

COROLLARY C. *Let $S^n \to E \to B$ be a fiber bundle with first Stiefel–Whitney class $w_1: \pi_1(B) \to \{\pm 1\}$. Then*

$$p_*\tau(E, E_A) = \tau(B, A) + (-1)^n \tau(B, A)^*$$

*where * denotes the involution of* $\mathrm{Wh}(\pi_1(B))$ *induced by* w_1.

Proof. Since $\pi_1(B)$ always acts trivially on $H_0(S^n; Z)$, $\sigma_{0*}\tau(B, A) = \tau(B, A)$ by the remarks preceding Corollary B. The action of $\pi_1(B)$ on $H_n(S^n; Z)$, however, is determined completely by $w_1: \pi_1(B) \to \{\pm 1\}$: if $w_1(\alpha) = 1$, then α preserves orientation; if $w_1(\alpha) = -1$, then α reverses orientation. In this case we show in 1.2 that $\sigma_{n*}\tau(B, A) = \tau(B, A)^*$.

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We note that since p_* is an isomorphism when $n > 1$, Corollary C gives a complete determination of $\tau(E, E_A)$ in terms of $\tau(B, A)$ and bundle invariants. It is also worth noting that when $w_1: \pi_1(B) \to \{\pm 1\}$ is trivial, then $\tau^* = \tau$ and Corollary C reduces to Corollary B.

There are some amusing special cases of Corollary C. In particular

COROLLARY D. *Let $S^n \to E \to B$ be a fiber bundle such that $\pi_1(B)$ is cyclic of order $2k$ and $w_1: \pi_1(B) \to \{\pm 1\}$ is non-trivial. Then $p_*\tau(E, E_A) \in \operatorname{Im}\{i_*: \mathrm{Wh}(Z_k) \to \mathrm{Wh}(Z_{2k})\}$. In particular if $k = 4$ or 6, then $p_*\tau(E, E_A) = 0$.*

Proof. Since $\pi_1(B) = Z_{2k}$, [1; p. 623] and [4; p. 359] show that the determinant induces an isomorphism $\det: K_1(Z(\pi_1(B)) \to U(Z(\pi_1(B)))$ where $U(Z(\pi_1(B))$ denotes the group of multiplicative units in $Z(\pi_1(B))$. Hence the determinant also induces an isomorphism $\det: \mathrm{Wh}(\pi_1(B)) \to U(Z(\pi_1(B)))/\pm\pi_1(B)$.

Let $t \in Z_{2k} \approx \pi_1(B)$ be a generator. Then every element $\lambda \in Z(\pi_1(B))$ may be regarded formally as a polynomial in t; that is $\lambda = \sum_{i=0}^{2k-1} a_i t^i$. Since $w_1: \pi_1(B) \to \{\pm 1\}$ is non-trivial, $w_1(t) = -1$, and $\lambda^* = \sum_{i=0}^{2k-1} (-1)^i a_i t^i$. In particular, if $\det \tau(B, A) = \sum_{i=0}^{2k-1} a_i t^i$ modulo $\pm\pi_1(B)$, then $\det \tau(B, A)^* = \sum_{i=0}^{2k-1} (-1)^i a_i t^i$. Hence

$$\begin{aligned} \det p_*\tau(E, E_A) &= \det(\tau(B, A) + (-1)^n \tau(B, A)^*) = (-1)^n \left(\sum_{i=0}^{2k-1} a_i t^i\right)\left(\sum_{i=0}^{2k-1} (-1)^i a_i t^i\right) \\ &= (-1)^n \left(\sum_{j=0}^{k-1} a_{2j} t^{2j} + \sum_{j=0}^{k-1} a_{2j+1} t^{2j+1}\right)\left(\sum_{j=0}^{k-1} a_{2j} t^{2j} - \sum_{j=0}^{k-1} a_{2j+1} t^{2j+1}\right) \\ &= (-1)^n \left[\left(\sum_{j=0}^{k-1} a_{2j} t^{2j}\right)^2 - \left(\sum_{j=0}^{k-1} a_{2j+1} t^{2j+1}\right)^2\right] = \sum_{j=0}^{k-1} b_j t^{2j} \end{aligned}$$

for some coefficients b_j. Letting $u \in Z_k$ be a generator such that $i(u) = t^2$ where $i: Z_k \to Z_{2k}$ is the inclusion, it follows that there is an element $\lambda = \sum_{j=0}^{k-1} b_j u^j$ such that $\det p_*\tau(E, E_A) = i_*\lambda$ modulo $\pm Z_{2k}$ where $i_*: Z(Z_k) \to Z(Z_{2k})$ is the inclusion.

But then $i_*\lambda$ must be a unit in $Z(Z_{2k})$ for a simple calculation shows that if $i_*\lambda = \sum_{i=0}^{k-1} b_i t^{2i}$ has multiplicative inverse $\sum_{i=0}^{2k-1} \beta_i t^i$, then $\sum_{i=0}^{k-1} \beta_{2i} t^i$ is also a multiplicative inverse for $i_*\lambda$. Hence $\mu = \sum_{i=0}^{k-1} \beta_{2i} u^i$ is a multiplicative inverse for λ. Therefore $\lambda \in U(Z_k)$ and represents an element $\bar{\lambda} \in \mathrm{Wh}(Z_k)$ such that $i_*\bar{\lambda} = p_*\tau(E, E_A)$.

When $k = 4$ or 6, $p_*\tau(B, A) = i_*\bar{\lambda} = 0$ since $\mathrm{Wh}(Z_k) = 0$ if $k = 4$ or 6.

In another direction, we have as an application of Corollaries B or C,

COROLLARY E. *Let $n \geq 1$ and $S^{2n+1} \to E \to M_1{}^m$ be an orientable bundle over the PL manifold M_1 of dimension $m \geq 2$. If $M_2{}^m$ is h-cobordant to M_1, then E fibers over M_2 with fiber S^{2n+1}.*

Proof. Let W be an h-cobordism between M_1 and M_2 and let $r: W \to M_1$ be a deformation retraction. Let $q: V \to W$ be the bundle over W induced from $p: E \to M_1$ by r. Then V is an h-cobordism between $E = q^{-1}(M_1)$ and $E' = q^{-1}(M_2)$. Since $q: V \to W$ has fiber S^{2n+1}, $n \geq 1$, $\tau(V, E) = \chi(S^{2n+1})\tau(W, M_1) = 0$. Thus V is PL homeomorphic to $E \times I$ and $E' = E$.

For certain groups π we can drop the unsightly hypothesis of Theorem A that $H_*(F; Z)$ be torsion free. In particular we prove

THEOREM F. *Let $F \to E \to B$ be a fiber bundle with connected fiber F and $\pi_1(B)$ cyclic. Then the action of $\pi_1(B)$ on $H_i(F; Z)$* /Torsion *determines an endomorphism σ_{i*} of* $\mathrm{Wh}(\pi_1(B))$ *and*

$$p_* \tau(E, E_A) = \sum (-1)^i \sigma_{i*} \tau(B, A).$$

As might be expected Theorems A and F are special cases of a more general theorem. We state and outline the proof of this theorem in Section 3.

Throughout this paper we work in the category $\mathscr{C}$ of polyhedra and piecewise linear (PL) maps [10]. In particular a *fiber bundle in $\mathscr{C}$*, or simply a *fiber bundle*, with fiber F is a map $p: E \to B$ in $\mathscr{C}$ such that there exist triangulations K and L of E and B respectively for which p is simplicial and such that for any simplex $\sigma \in L$, there is a homeomorphism (in $\mathscr{C}$) $h: \sigma \times F \to p^{-1}(\sigma)$ satisfying $ph = p_1$ where $p_1: \sigma \times F \to \sigma$ is projection on the first factor.

It is a straightforward, but tedious, matter to verify that all the standard results of fiber bundle theory hold for fiber bundles in $\mathscr{C}$. In particular we can form induced bundles, and homotopic maps induce equivalent fiber bundles.

This paper is organized as follows. In Section 1 we develop a general method for constructing endomorphisms of Whitehead groups of which the σ_{i*} of Theorems A and F are special cases. The general problem is reduced to a special case in Section 2, and in Section 3 the analysis of the special case is begun. Sections 4 and 5 complete the more technical aspects of the analysis of the special case and an appendix sketches the proof of a technical, but elementary, lemma.

An earlier draft of this paper explored this problem only for orientable bundles. The author would like to thank D. S. Kahn for suggesting that the same approach should apply to non-orientable bundles as well. The author would also like to thank M. Cohen for pointing out a technical mistake in the earlier draft.

The referee has pointed out that Theorem A may be reformulated as follows: Let $G_R(\pi)$ denote the representation ring of π over R. The essential content of Proposition 1.1 is that $K_1(R(\pi))$ is a module over $G_R(\pi)$ and that $\mathrm{Wh}(\pi)$ is a module over $G_Z(\pi)$. (In this lemma we consider anti-representations, rather than representations, of π. This may be avoided by considering left, rather than right, $R(\pi)$ modules.)

Now let $F \to E \to B$ be a fiber bundle with $H_*(F; Z)$ torsion free and let $\chi_{\pi_1(B)}(F; Z) \in G_R(\pi_1(B))$ be the alternating sum of the representations of $\pi_1(B)$ on $H_i(F; Z)$ induced by the "action" of $\pi_1(B)$ on F. Then $\chi_{\pi_1(B)}(F; Z)$ is a kind of Euler characteristic, and the formula of Theorem A may be written as

$$p_* \tau(E, E_A) = \chi_{\pi_1(B)}(F; Z) \tau(B, A).$$

Added in Proof: Since writing this paper, the author has discovered that many of the results of §1 are well known (cf. R. G. Swan: Induced Representations and Projective Modules, Ann. Math. **71** (1960), 552–578; or [1, pp. 558–569]).

§1. SOME ENDOMORPHISMS OF THE WHITEHEAD GROUP

Let R be a commutative ring with unit and R^n denote the direct sum of n copies of R. It is the object of this section to prove

PROPOSITION 1.1. (i) *Let $\sigma : \pi \to \mathrm{Aut}_R(R^n)$ be an anti-homomorphism of the group π into the automorphisms of R^n. Then σ induces an endomorphism $\bar{\sigma}_* : K_1(R(\pi)) \to K_1(R(\pi))$.*

(ii) *If $\sigma : \pi \to \mathrm{Sim\ Aut}_R(R^n)$, the simple automorphisms of R^n, then σ induces an endomorphism $\sigma_* : K_1(R(\pi))/\pm\pi \to K_1(R(\pi))/\pm\pi$.*

An automorphism of R^n is *simple* if it goes to zero in $\bar{K}_1(R)$.

In some cases the homomorphisms σ_* can be evaluated. In particular we have

PROPOSITION 1.2. (i) *If σ is the trivial anti-homomorphism, then σ_* is multiplication by n.*

(ii) *If $R = Z$ and $n = 1$, σ_* is the involution of* $\mathrm{Wh}(\pi)$ *induced by the involution of* $Z(\pi)$ *that sends $\sum_{\alpha\in\pi} n_\alpha \alpha$ to $\sum_{\alpha\in\pi} n_\alpha \sigma(\alpha)\alpha$ where* $\mathrm{Aut}_Z(Z)$ *has been identified with* $Z_2 = \{\pm 1\}$.

We note that the involution in (ii) above is not the usual duality involution as α goes to $\sigma(\alpha)\alpha$ not $\sigma(\alpha)\alpha^{-1}$.

Before proving 1.1 and 1.2, we recall the definition of the functor K_1 from a categorical point of view (cf. [2; pp. 31–32]). Let A be any ring and $\mathscr{P}$ be the category of finitely generated projective right A modules and A homomorphisms. Let $\mathscr{P}[T]$ be the category whose objects are pairs (P, f) where P is an object in $\mathscr{P}$ and $f \in \mathrm{Aut}_A(P)$. A morphism $i : (P_1, f_1) \to (P_2, f_2)$ is an A homomorphism such that $f_2 i = i f_1$. Then $K_1(A)$ is the abelian group generated by the isomorphism classes of objects from $\mathscr{P}[T]$ with the relations

(a) If $0 \to (P_2, f_2) \to (P_1, f_1) \to (P_0, f_0) \to 0$ is exact in $\mathscr{P}[T]$, then $[P_1, f_1] \sim [P_2 + P_0, f_2 + f_0]$, and

(b) $[P, gf] \sim [P, g] + [P, f]$

where the square bracket denotes the isomorphism class.

Returning now to the context of 1.1, let $\mathscr{P}$ be the category of finitely generated right projectives over $R(\pi)$ and let P be an object of $\mathscr{P}$. Let $P \otimes_\sigma R^n$ be the right $R(\pi)$ module that additively is $P \otimes_R R^n$ and whose scalar multiplication is defined by setting $(x \otimes y)\alpha = x\alpha \otimes \sigma(\alpha)(y)$ for $\alpha \in \pi$ and extending linearly. If $f : P_1 \to P_2$ is an $R(\pi)$ homomorphism, then so is $f \otimes 1 : P_1 \otimes_\sigma R^n \to P_2 \otimes_\sigma R^n$, for $(f \otimes 1)[x \otimes y)\lambda] = (f \otimes 1)(x\lambda \otimes \sum r_\alpha \sigma(\alpha)(y)) = f(x)\lambda \otimes \sum r_\alpha \sigma(\alpha)(y) = [f(x) \otimes y]\lambda = [(f \otimes 1)(x \otimes y)]\lambda$.

LEMMA 1.3. *If $P \in \mathscr{P}$, then $P \otimes_\sigma R^n \in \mathscr{P}$.*

Proof. Suppose first that P is free and let $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ be $R(\pi)$ and R bases respectively for P and R^n. Let $x \otimes y \in P \otimes_\sigma R^n$. Then there exist $\lambda_i \in R(\pi)$ and $r_j \in R$ such that $x = \sum_i e_i \lambda_i$ and $y = \sum_j r_j f_j$. Also $\lambda_i = \sum_\alpha s_{i,\alpha} \alpha$ where for fixed i, $s_{i,\alpha} \neq 0$ for only finitely many α. Then

$$
\begin{aligned}
x \otimes y &= \sum_{i,j} e_i \lambda_i \otimes r_j f_j = \sum_{i,j} e_i \sum_\alpha s_{i,\alpha} \alpha \otimes r_j f_j = \sum_{i,j,\alpha} e_i s_{i,\alpha} \alpha \otimes r_j f_j \\
&= \sum_{i,j,\alpha} (e_i \otimes \sigma(\alpha)^{-1} f_j) s_{i,\alpha} r_j \alpha = \sum_{i,j,\alpha} [e_i \otimes (\sum_k t_{j,k}(\alpha) f_k] s_{i,\alpha} r_j \alpha \\
&= \sum_{i,j,k,\alpha} (e_i \otimes f_k) t_{j,k}(\alpha) s_{i,\alpha} r_j \alpha = \sum_{i,k} (e_i \otimes f_k) \sum_{j,\alpha} t_{j,k}(\alpha) s_{i,\alpha} r_j \alpha
\end{aligned}
$$

where $(t_{j,k}(\alpha))$ is the matrix of $\sigma(\alpha)^{-1}$ relative to $f_1, \ldots, f_n$. But now since there are only finitely many i, and since for fixed i, $s_{i,\alpha} \neq 0$ for only finitely many α, $\sum_j t_{j,k}(\alpha) s_{i,\alpha} r_j \neq 0$ for only finitely many α. Hence $\sum_\alpha (\sum_j t_{j,k}(\alpha) s_{i,\alpha} r_j)\alpha \in R(\pi)$ and $\{e_i \otimes f_k \mid i = 1, \ldots, m;\ k = 1, \ldots, n\}$ generates $P \otimes_\sigma R^m$ over $R(\pi)$.

Suppose $\sum_{i,j}(e_i \otimes f_j)\lambda_{i,j} = 0$ for some $\lambda_{i,j} \in R(\pi)$. Then writing $\lambda_{i,j} = \sum_\alpha r_{i,j,\alpha}\alpha$ where only finitely many $r_{i,j,\alpha} \neq 0$ for fixed i, j, we would have $0 = \sum_{i,j}(e_i \otimes f_j)\sum_\alpha r_{i,j,\alpha}\alpha = \sum_{i,j,\alpha}(e_i\alpha \otimes \sigma(\alpha)f_j)r_{i,j,\alpha} = \sum_{i,j,\alpha}[e_i\alpha \otimes (\sum s_{j,k}(\alpha)f_k)]r_{i,j,\alpha} = \sum_{i,j,k,\alpha}(e_i\alpha \otimes f_k)r_{i,j,\alpha}s_{j,k}(\alpha)$. Since $\{e_i\alpha \otimes f_k | i = 1, \ldots, m; k = 1, \ldots, n; \alpha \in \pi\}$ is an R basis for $P \otimes_\sigma R^m$, it follows that the coefficients of $e_i\alpha \otimes f_k$ in the above expression must all vanish. Hence $\sum_j r_{i,j,\alpha}s_{j,k}(\alpha) = 0$ for all i, k, α. This means that the matrix product $(r_{i,j,\alpha})(s_{j,k}(\alpha))$ vanishes for all α. Since $(s_{j,k}(\alpha))$ is invertible, this implies that $(r_{i,j,\alpha}) = 0$. Hence $\lambda_{i,j} = \sum r_{i,j,\alpha}\alpha = 0$ and $\{e_i \otimes f_j\}$ is a basis for the free module $P \otimes_\sigma R^m$.

Suppose now that P is an arbitrary object of $\mathscr{P}$. Then there is a finitely generated free $R(\pi)$ module F such that P is a direct summand of F. Since it is easily seen that $P \otimes_\sigma R^n$ is then a direct summand of $F \otimes_\sigma R^n$ and $F \otimes_\sigma R^n$ is finitely generated and free over $R(\pi)$ by the discussion above, $P \otimes_\sigma R^n \in \mathscr{P}$ and the lemma is proved.

Proof of 1.1 (i): Define a functor $\sigma_* : \mathscr{P}[T] \to \mathscr{P}[T]$ by $\sigma_*(P, f) = (P \otimes_\sigma R^n, f \otimes 1)$. Since all modules under consideration are projective, σ_* preserves exact sequences. Also $\sigma_*(P, gf) = (P \otimes_\sigma R^n, gf \otimes 1) = (P \otimes_\sigma R^n, (g \otimes 1)(f \otimes 1))$. Hence σ_* carries relations of types (a) and (b) into relations of types (a) and (b), and σ_* induces a homomorphism $\bar{\sigma}_* : K_1(R(\pi)) \to K_1(R(\pi))$. The first part of 1.1 follows.

In order to prove 1.1 (ii) and 1.2, it is necessary to interpret $\bar{\sigma}_*$ in terms of the usual matrix definition of $K_1(R(\pi))$. Specifically if $M \in GL(m, R(\pi))$ is a matrix representing $x \in K_1(R(\pi))$, we wish to describe a matrix representing $\bar{\sigma}_*(x)$. This can be done as follows: let $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ be $R(\pi)$ and R bases for $R(\pi)^m$ and R^n respectively. By the proof of 1.3, $\{e_i \otimes f_j | i = 1, \ldots, m; j = 1, \ldots, n\}$ is an $R(\pi)$ basis for $R(\pi)^m \otimes_\sigma R^n$. Relative to these bases, M determines an object $(R(\pi)^m, f)$ which also represents x and $\bar{\sigma}_*(x)$ is represented by the matrix of $f \otimes 1 : R(\pi)^m \otimes_\sigma (R^n \to R(\pi)^m \otimes_\sigma R^n$. With this interpretation in mind, we give the

Proof of 1.1 (ii): Let $x \in K_1(R(\pi))$ be represented by $(\alpha) \in GL(1, R(\pi))$ where $\alpha \in \pi$. In order to show that $\bar{\sigma}_*$ induces an endomorphism of $K_1(R(\pi))/\pm\pi$ it suffices to show that $\bar{\sigma}_*(x) = 0$ in $K_1(R(\pi))/\pm\pi$. By the interpretation above, however, $\sigma_*(x)$ is represented by the matrix of $f \otimes 1 : R(\pi) \otimes_\sigma R^n \to R(\pi) \otimes_\sigma R^n$ where the matrix of f is (α). But then $(f \otimes 1)(e_1 \otimes f_j) = e_1\alpha \otimes f_j = (e_1 \otimes \sigma(\alpha)^{-1}f_j)\alpha = (e_1 \otimes \sum_i s_{ji}f_j)\alpha = \sum_i (e_1 \otimes f_i)s_{ji}\alpha$ where (s_{ji}) is the matrix of $\sigma(\alpha)^{-1}$. Therefore $\sigma_*(x)$ is represented by $(s_{ji}\alpha) = (s_{j,i})$ diag$(\alpha, \ldots, \alpha)$. Clearly diag$(\alpha, \ldots, \alpha)$ goes to zero in $K_1(R(\pi))/\pm\pi$, and since $\sigma : \pi \to \text{Sim Aut}_R(R^n)$, $(s_{j,i})$ also goes to zero. Hence $\bar{\sigma}_*(x)$ goes to zero in $K_1(R(\pi))/\pm\pi$ and 1.1 (ii) follows.

Proof of 1.2: Let $\sigma : \pi \to \text{Aut}_R(R^n)$ be the trivial anti-homomorphism and let $x \in K_1(R(\pi))/\pm\pi$ be represented by $M = (\lambda_{kl}) \in GL(m, R(\pi))$. Then $\sigma_*(x)$ is represented by the matrix of $f \otimes 1$ relative to the basis $\{e_i \otimes f_j | i = 1, \ldots, m; j = 1, \ldots, n\}$ of $R(\pi)^m \otimes_\sigma R^n$ where $f: R(\pi)^m \to R(\pi)^m$ has matrix M. But $(f \otimes 1)(e_i \otimes f_j) = f(e_i) \otimes f_j = (\sum_k e_k\lambda_{ki}) \otimes f_j = \sum_k(e_k \otimes f_j)\lambda_{ki}$ since σ is trivial, and if the basis of $R(\pi)^m \otimes R^n$ is ordered $e_1 \otimes f_1, \ldots, e_m \otimes f_1, e_1 \otimes f_2, \ldots, e_m \otimes f_2, e_1 \otimes f_3, \ldots, e_m \otimes f_n$ the matrix of $f \otimes 1$ is just diag$(M, \ldots, M)$ with n copies of M. Hence $\sigma_*(x) = nx$.

Suppose that $\sigma : \pi \to \text{Aut}_Z(Z) = \{\pm 1\}$ and let $x \in \text{Wh}(\pi) = K_1(Z(\pi))/\pm\pi$ be represented by $M = (\lambda_{i,j}) \in GL(m, Z(\pi))$. Then $\sigma_*(x)$ is represented by the matrix of $f \otimes 1 : Z(\pi)^m \otimes_\sigma Z \to Z(\pi)^m \otimes_\sigma Z$ where f has matrix M. Now $(f \otimes 1)(e_i \otimes f_1) = f(e_i) \otimes f_1 = \sum_k (e_k \lambda_{ki} \otimes f_1)$ and $e_k \lambda_{ki} \otimes f_1 = (e_k \sum_{\alpha\in\pi} n_{ki\alpha} \alpha) \otimes f_1 = \sum_{\alpha\in\pi} [(e_k \otimes \sigma(\alpha)^{-1} f_1) n_{ki\alpha} \alpha]$ where $n_{ki\alpha} \in Z$. But $\sigma(\alpha)^{-1} = \sigma(\alpha) = \pm 1$, and $e_k \otimes \sigma(\alpha)^{-1} f_1 = (e_k \otimes f_1)\sigma(\alpha)$. Therefore $\sum_{\alpha\in\pi} [(e_k \otimes \sigma(\alpha)^{-1} f_1) n_{ki\alpha} \alpha] = (e_k \otimes f_1) \sum_{\alpha\in\pi} n_{ki\alpha} \sigma(\alpha)\alpha$ and the matrix of $f \otimes 1$ is obtained by replacing $\lambda_{ki} = \sum_\alpha n_{ki\alpha} \alpha$ by $\sum_\alpha n_{ki\alpha} \sigma(\alpha)\alpha$. Part (ii) of 1.2 follows.

The endomorphisms $\bar\sigma_*$ and σ_* are natural under changes in the ground ring in the following sense. Let S be a commutative ring with unit and $\rho : R \to S$ be a ring homomorphism. Then S becomes a left R module by setting $rs = \rho(r)s$ and ρ induces a homomorphism $\rho_* : \mathrm{Aut}_R(R^n) \to \mathrm{Aut}_S(S^n)$ obtained by sending an automorphism f of R^n to the automorphism $f \otimes 1$ of $R^n \otimes_R S$ and identifying $R^n \otimes_R S$ with S^n. Let $\tau = \rho_* \sigma : \pi \to \mathrm{Aut}_S(S^n)$ and let ρ_* denote ambiguously the maps $\rho_* : K_1(R(\pi)) \to K_1(S(\pi))$ and $\rho_* : K_1(R(\pi))/\pm\pi \to K_1(S(\pi))/\pm\pi$ induced by ρ.

LEMMA 1.4. *With the notation above, $\rho_* \bar\sigma_* = \bar\tau_* \rho_*$. In addition, if $\sigma : \pi \to \mathrm{Sim\ Aut}_R(R^n)$, then $\tau : \pi \to \mathrm{Sim\ Aut}_S(S^n)$, and $\rho_* \sigma_* = \tau_* \rho_*$.*

Proof. Let $x \in K_1(R(\pi))$ be represented by $(P, f) \in \mathscr{P}[T]$. Then $\rho_* \bar\sigma_*(x)$ is represented by $\rho_*(P \otimes_\sigma R^n, f \otimes 1_{R^n}) = (P \otimes_\sigma R^n \otimes_R S, f \otimes 1_{R^n} \otimes 1_S)$, while $\bar\tau_* \rho_*(x)$ is represented by $\tau_*(P \otimes_R S,\ f \otimes 1_S) = (P \otimes_R S \otimes_\tau S^n,\ f \otimes 1_S \otimes 1_{S^n})$. Now additively $P \otimes_\sigma R^n \otimes_R S = P \otimes_R R^n \otimes_R S \approx P \otimes_R S^n \approx P \otimes_R S \otimes_S S^n = P \otimes_R S \otimes_\tau S^n$ and clearly $f \otimes 1_{R^n} \otimes 1_S$ corresponds to $f \otimes 1_S \otimes 1_{S^n}$. The definition of τ shows immediately that $P \otimes_\sigma R^n \otimes_R S$ is isomorphic to $P \otimes_R S \otimes_\tau S^n$ as $S(\pi)$ modules. Hence $\rho\bar\sigma_*(x) = \bar\tau_* \rho_*(x)$.

If $\sigma : \pi \to \mathrm{Sim\ Aut}_R(R^n)$, then $\tau : \pi \to \mathrm{Sim\ Aut}_S(S^n)$ since ρ_* preserves simple automorphisms. The last clause of 1.4, then follows from the first.

§2. A REDUCTION OF THE PROBLEM TO A SPECIAL CASE

It is the object of this section to reduce the problem of computing $p_* \tau(E, E_A)$ to the special case when A is a manifold of dimension greater than 5 (possibly with boundary) and B is an h-cobordism with one end A. There are two steps in this reduction; the first replaces the original pair (B, A) by a new pair (B', M) where M is a manifold of dimension greater than 5, and the second replaces B' by an h-cobordism.

The key lemma used in the reduction of the problem to the special case is

LEMMA 2.1. *Let B be obtained from A by an elementary formal expansion and let $p : E \to B$ be a fiber bundle over B. Then $\tau(E, E_A) = 0$.*

For the definitions of expansions and contractions see [8], [9], or [10].

Proof. Let (K, L) be a triangulation of (B, A) such that $K = L + S + vS$ for some simplex S. To show $\tau(E, E_A) = 0$, it suffices to show that there is a formal deformation $D : p^{-1}(vS) \to p^{-1}(v\dot S)$ relative to $p^{-1}(v\dot S)$; for then certainly there is a formal deformation $D' : E \to E_A$ relative to E_A and the torsion may be regarded as the "obstruction" to finding such a deformation (cf. for example [7; p. 63]).

Since $p : E \to B$ is *a* fiber bundle, there is a PL homeomorphism $h : vS \times F \to p^{-1}(vS)$

which commutes with projection; i.e. $ph = p_1$ where $p_1 : vS \times F \to vS$ is projection on the first factor. Since vS collapses to $v\dot{S}$, the Product Theorem for Whitehead Torsion [3; Corollary 1.3] shows that $\tau(vS \times F, v\dot{S} \times F) = 0$. Thus $vS \times F$ deforms formally to $v\dot{S} \times F$ relative to $v\dot{S} \times F$. Hence $h(vS \times F) = p^{-1}(vS)$ deforms formally to $h(v\dot{S} \times F) = p^{-1}(v\dot{S})$ relative to $p^{-1}(v\dot{S})$ and the lemma follows.

COROLLARY 2.2. *Let B be an expansion of A and let $p : E \to B$ be a fiber bundle over B. Then $\tau(E, E_A) = 0$.*

Proof. Let (K, L) be a triangulation of (B, A) such that there are subcomplexes K_i of K, $i = 1, \ldots, n$ such that $K_0 = L$, $K_n = K$ and K_{i+1} is an elementary expansion of K_i. Let $E_i = p^{-1}(|K_i|)$. Then $\tau(E, E_A) = \sum_{i=1}^{n} k_{i*}\tau(E_i, E_{i-1})$ where $k_i : |K_i| \to |K| = B$ is the inclusion. Since $\tau(E_i, E_{i-1}) = 0$ by the lemma, the corollary follows.

Now embed A in R^n for some $n \geq 5$ and let M^n be a regular neighborhood of A. Let $B' = M \cup B$. Since M contracts to A, B' contracts to B and there are deformation retractions $r : M \to A$ and $r' = r \cup 1_B : B' \to B$ where 1_B denotes the identity map of B. Let $p' : E' \to B'$ be induced from $p : E \to B$ by r' and let $E_M = p'^{-1}(M)$. Since $r'|B = 1_B$, we can, and do, identify $p'^{-1}(B)$ with E.

LEMMA 2.3. *E_M is a deformation retract of E' and $g_*\tau(E', E_M) = \tau(E, E_A)$ where $g : E' \to E$ is a bundle map covering r'.*

Proof. The first part of the lemma follows by noting that M is a deformation retract of B' and applying the Covering Homotopy Theorem. To see the statement about the torsions consider the diagram

$$\begin{array}{ccc} E & \overset{i}{\subset} & E' \\ \cup & & \cup_j \\ E_A & \subset & E_M \end{array}$$

and note that all the inclusions are homotopy equivalence. Then $i_*\tau(E, E_A) + \tau(E', E) = \tau(E', E_A) = j_*\tau(E_M, E_A) + \tau(E', E_M)$. By Corollary 1.2 $\tau(E', E) = 0 = \tau(E_M, E_A)$ since B' and A' are expansions of B and A respectively. Therefore $i_*\tau(E, E_A) = \tau(E', E_M)$. The conclusion now follows by applying g_* and noting that since $gi = 1_E$, g_*i_* is the identity.

LEMMA 2.4. *There is a manifold W^{n+1} containing M^n as a deformation retract and triangulations, J, K, and L of W, B', and M respectively such that there is a formal deformation $D : J \to K$ relative to L.*

Proof. Let $r' : B' \to M$ be a deformation retraction and let W^{n+1} be the h-cobordism satisfying $j_*^{-1}\tau(W, M) = r_*'\tau(B', M)$ where $r_*' : \mathrm{Wh}(\pi_1(B')) \to \mathrm{Wh}(\pi_1(M))$ is induced by r' and $j_* : \mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}(\pi_1(W))$ is induced by the inclusion. Since both r' and j are homotopy equivalences, so is jr'. Furthermore $\tau(jr') = j_*\tau(r') + \tau(j) = j_*\tau(r') + \tau(W, M)$ and $0 = \tau(1_M) = \tau(r'i) = r_*'\tau(i) + \tau(r') = r_*'\tau(B', M) + \tau(r')$ where 1_M denotes the identity of M and $i : M \subset B'$. Therefore $\tau(jr') = -j_*r_*'\tau(B', M) + \tau(W, M) = 0$ by the choice of W and the lemma follows from [9; Theorem 13] and its simplicial analogues [8].

Let $f : W \to B'$ be the map associated to D. Since D determines f up to homotopy, the bundle $q : E_W \to W$ induced from $p' : E' \to B'$ by f is uniquely determined by D. We note that since D is relative to M, $f|M = 1_M$; hence $q^{-1}(M) = E_M$.

LEMMA 2.5. *$\tau(E', E_M) = h_* \tau(E_W, E_M)$ where $h_* : \mathrm{Wh}(\pi_1(E_W)) \to \mathrm{Wh}(\pi_1(E'))$ is induced by a bundle map h covering f.*

Proof. Let W and B' be as above. Then by [8; Theorem 5, addendum 2], there is a complex K'' and subcomplexes K, K', L such that (K', L) triangulates $(B' M)$, (K, L) triangulates (W, M) and K'' contracts to both K and K'. Let B'' be the space underlying K'' and let $f' : B'' \to B'$ be associated with the contraction $C : K'' \to K'$. Finally let $p'' : E'' \to B''$ be the bundle induced from $p' : E' \to B'$ by f' and let $g' : E'' \to E'$ cover f'. We note that since the composite $W \subset B'' \xrightarrow{f'} B'$ is associated to the formal deformation $K \xrightarrow{E} K'' \xrightarrow{C} K'$, $f'|W$ is homotopic to f and $g'|E_W$ is homotopic to g.

Since W and B' are deformation retracts of B'' and M is a deformation retract of W and B', each of the inclusions in the diagram

$$\begin{array}{ccc} E_W & \overset{i}{\subset} & E'' \\ \cup & & \cup\, j \\ E_M & \subset & E' \end{array}$$

is a homotopy equivalence. Thus $\tau(E'', E_W) + i_* \tau(E_W, E_M) = \tau(E'', E_M) = \tau(E'', E') + j_* \tau(E' E_M)$ and $i_* \tau(E_W, E_M) = j_* \tau(E', E_M)$ since the other torsions vanish by 2.2. By applying g_*' to both sides of the equation, the lemma follows.

PROPOSITION 2.6. *If Theorems* A *and* F *hold when B is an h-cobordism from A to another manifold, then they hold in general.*

Proof. Let (B, A) be an arbitrary pair with A a deformation retract of B. By the construction given above we obtain a manifold pair (W, M) and a commutative diagram

$$\begin{tikzcd} Ew \arrow[r, "gh"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ W \arrow[r, "r'f"] & B. \end{tikzcd}$$

Therefore if $q_* \tau(E_W, E_M) = \sum (-1)^i \sigma_{i*} \tau(W, M)$ then $p_* \tau(E, E_A) = p_* g_* h_* \tau(E_W, E_M) = r_*' f_* q_* \tau(E_W, E_M) = \sum (-1)^i \sigma_{i*} r_*' f_* \tau(W, M)$ by 2.3 and 2.5. From the commutative diagram of homotopy equivalences

$$\begin{tikzcd} M \arrow[r, "r'f|M"] \arrow[d, "i"'] & A \arrow[d, "j"] \\ W \arrow[r, "r'f"] & B \end{tikzcd}$$

however, we see that $r_*' f_* \tau(W, M) + \tau(r'f) = \tau(r'fi) = \tau(jr'f|M) = j_* \tau(r'f|M) + \tau(B, A)$.

Since $r'f$ and $r'f|M$ are both simple homotopy equivalences, $r_*'f_*\tau(W, M) = \tau(B, A)$ and the proof is complete.

§3. THE ANALYSIS OF THE SPECIAL CASE

In this section we analyze the special case of the problem and outline the proof of the theorem from which Theorems A and F follow. We first fix some notation.

Let (X, A) be a polyhedral pair, π be a discrete group, and $q : \hat{X} \to X$ be a principal π bundle where π acts on $\hat{X}$ from the right. Let $\hat{A} = q^{-1}(A)$. Let (K, L) be a triangulation of (X, A) and $(\hat{K}, \hat{L})$ be the triangulation of $(\hat{X}, \hat{A})$ that covers (K, L). Then $C_*(\hat{K}, \hat{L})$ denotes the cellular chain complex of $(\hat{K}, \hat{L})$ regarded as a free right module ober $Z(\pi)$ and endowed with a family of preferred bases consisting of one simplex $\bar{\sigma} \in \hat{K} - \hat{L}$ for each $\sigma \in K - L$. For any commutative ring R then $C_*(\hat{K}, \hat{L}) \otimes R$ is free over $R(\pi)$ with a preferred basis. If $H_*(\hat{K}, \hat{L}; R)$ is also free over $R(\pi)$ and has a preferred basis, then the torsion of $C_*(\hat{K}, \hat{L}) \otimes R$ is defined [4; p. 365] and we shall denote this torsion by $\Delta_R(X, A; \pi)$. It is well known that $\Delta_R(X, A; \pi)$ does not depend on the choice of the triangulation (K, L). When the principal π bundle $q : \hat{X} \to X$ is clear from the context and no confusion arises we shall write $\Delta_R(X, A)$ instead of $\Delta_R(X, A; \pi)$. In particular when $q : \hat{X} \to X$ is the universal cover of X, we write $\Delta_R(X, A)$ instead of $\Delta_R(X, A; \pi_1(X))$.

Let $p : E \to B$ be a fiber bundle with fiber F, and recall that there is an anti-homomorphism $\bar{\sigma} : \pi_1(B) \to \mathrm{Iso}(F)$ where $\mathrm{Iso}(F)$ denotes the group of isotopy classes of homeomorphisms of F onto itself (cf. [5; p. 101]). The induced anti-homomorphism $\sigma_i : \pi_1(B) \to \mathrm{Aut}_R(H_i(F; R))$ defined by $\sigma(\alpha) = \bar{\sigma}(\alpha)_*$ for $\alpha \in \pi_1(B)$ is called the *action* of $\pi_1(B)$ on $H_i(F, R)$. If $\sigma_i(\alpha)$ goes to zero in $\bar{K}_1(R)$ for every α, the action is called *simple*. If σ_i is simple, let $\sigma_{i*} : \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(B))$ be the endomorphism of 1.1.

It is the object of this section to outline the proof of:

THEOREM 3.1. *Let $n \geq 5$, (W^{n+1}, M^n, N^n) be an h-cobordism between M and N, and $p : E \to W$ be a fiber bundle with connected fiber F. Let R be a principal ideal domain. If $H_*(F; R)$ is free over R and the action of $\pi_1(W)$ on $H_i(F; R)$ is simple for every i, then this action induces endomorphisms σ_{i*} of $K_1(R(\pi_1(W)))/\pm\pi_1(W)$ and*

$$p_*\Delta_R(E, E_M) = \sum (-1)^i \sigma_{i*} \Delta_R(W, M)$$

where $E_M = p^{-1}(M)$.

Theorem A follows from 3.1 by letting $R = Z$ and recalling that since $\bar{K}_1(Z) = 0$, the action of $\pi_1(W)$ on $H_i(F; Z)$ is always simple. Now let $R = Q$, the rationals, and $\rho : Z \to Q$ be the inclusion. Since $H_*(F; Q) = \{H_*(F; Z)/\mathrm{Torsion}\} \otimes Q$, $\sigma_i : \pi_1(W) \to \mathrm{Aut}_Q(H_i(F, Q))$, factors as $\rho_*\sigma_i'$ where $\sigma_i' : \pi_1(W) \to \mathrm{Aut}_Z(H_i(F; Z)/\mathrm{Torsion})$ and the action of $\pi_1(W)$ on $H_i(F; Q)$ is simple. More importantly, however, σ_i' induces an endomorphism σ_{i*} of $\mathrm{Wh}(\pi_1(W))$ and $\rho_*\sigma_{i*} = \sigma_{i*}'\rho_*$ by 1.4. Then $\rho_* p_* \tau(E, E_M) = p_*\rho_*\tau(E, E_M) = p_*\Delta_Q(E, E_M) = \sum(-1)^i\sigma_{i*}\Delta_Q(W, M) = \sum(-1)^i\sigma_{i*}\rho_*\tau(W, M) = \sum(-1)^i\rho_*\sigma_{i*}\tau(W, M) = \rho_*\sum(-1)^i\sigma_{i*}'\tau(W, M)$ and Theorem F follows from

LEMMA 3.2. *Let $\rho : Z \to Q$ be the inclusion and let π be a cyclic group. Then $\rho_* : \mathrm{Wh}(\pi) \to K_1(Q(\pi))/\pm\pi$ is a monomorphism.*

Proof. If π is infinite cyclic the result is obvious since then $\mathrm{Wh}(\pi) = 0$. So suppose π is finite cyclic and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & SK_1(Z(\pi)) \arrow[r] \arrow[d] & K_1(Z(\pi)) \arrow[r, "\det"] \arrow[d, "\rho_*"] & U(Z(\pi)) \arrow[r] \arrow[d, "\bar{\rho}_*"] & 0 \\
0 \arrow[r] & SK_1(Q(\pi)) \arrow[r] & K_1(Q(\pi)) \arrow[r, "\det"] & U(Q(\pi)) \arrow[r] & 0
\end{tikzcd}$$

where $U(R)$ denotes the group of multiplicative units of R, $R = Z(\pi)$, $Q(\pi)$, and det is the homomorphism induced by taking determinants. Now for π finite cyclic, $SK_1(Z(\pi)) = 0$ [1; p. 623] while $\bar{\rho}_*$ is clearly monomorphic. Thus ρ_* is monomorphic and since $\rho_*(\pm\pi) = \pm\pi \subset K_1(Q(\pi))$, $\rho_* : \mathrm{Wh}(\pi) \to K_1(Q(\pi))/\pm\pi$ is also monomorphic.

The proof of Theorem 3.1 requires two lemmas.

LEMMA 3.3. *Let C_* be a based acyclic chain complex of right modules over R and $\rho : R \to R'$ be a ring homomorphism. Then $C_* \otimes_\rho R'$ is a based acyclic chain complex over R' and $\rho_* \Delta(C_*) = \Delta(C_* \otimes_\rho R')$ where $\Delta(\ \ )$ denotes the torsion of the chain complex.*

Remark. $C_* \otimes_\rho R'$ is the chain complex obtained by regarding R' as a right R module via σ (i.e. $r \cdot r' = \rho(r)r'$) and tensoring over R.

Proof. Since C_* is acyclic, there is a contracting homotopy $\delta : C_* \to C_*$ such that $\delta^2 = 0$ and by definition $\Delta(C_*)$ is represented (up to sign) by the matrix of $(\partial + \delta) : \sum C_{2i} \to \sum C_{2i+1}$ (cf. [7; p. 43]). But then $\delta \otimes 1$ is a contracting homotopy of $C_* \otimes_\rho R'$ and it is easy to see that the matrix of $(\partial \otimes 1 + \delta \otimes 1) : \sum C_{2i} \otimes_\rho R' \to \sum C_{2i+1} \otimes_\rho R'$ is just the image of the matrix of $(\partial + \delta)$ under σ_*. Hence $\rho_* \Delta(C_*) = \Delta(C_* \otimes_\rho R')$.

LEMMA 3.4. *Let (K, L) be a simplicial pair such that $i_* : \pi_1(L) \to \pi_1(K)$ is an isomorphism and let $\rho : \pi_1(K) \to G$ be an epimorphism. Then there is a simple isomorphism $C_*(\tilde{K}, \tilde{L}) \otimes_\rho Z(G) \to C_*(\hat{K}, \hat{L})$ where $\tilde{K}$ is the universal covering space of K, $\hat{K}$ is the regular covering space of K corresponding to the kernel of ρ, and $\pi_1(K)/\ker \rho$ has been identified with G via ρ.*

Proof. Since $\hat{K}$ is a covering space of K and $\tilde{K}$ is the universal cover, $\tilde{K}$ is also the universal cover of $\hat{K}$. Let $q : \tilde{K} \to \hat{K}$ be the covering map and note that q is simplicial. Let $\tilde{k}_0 \in \tilde{K}$ and $\hat{k}_0 \in \hat{K}$ be base points such that $q(\tilde{k}_0) = \hat{k}_0$. Then under the usual identifications of $\pi_1(K)$ and $\pi_1(K)/\ker \rho$ with the groups of covering transformations of $\tilde{K}$ and $\hat{K}$ respectively and the identification of $\pi_1(K)/\ker \rho$ with G via ρ, there is a commutative diagram

$$\begin{tikzcd}
\tilde{K} \arrow[r, "\alpha"] \arrow[d, "q"'] & \tilde{K} \arrow[d, "q"] \\
\hat{K} \arrow[r, "\rho(\alpha)"] & \hat{K}
\end{tikzcd}$$

for any $\alpha \in \pi_1(K)$. Then $q_* : C_*(\tilde{K}, \tilde{L}) \to C_*(\hat{K}, \hat{L})$ satisfies $q_*(c\lambda) = (q_* c)\rho(\lambda)$ for any $c \in C_*(\tilde{K}, \tilde{L})$ and $\lambda \in Z(\pi_1(K))$. The map $h : C_*(\tilde{K}, \tilde{L}) \otimes_\rho Z(G) \to C_*(,\hat{K}\ \hat{L})$ given by $h(c \otimes \mu) = (q_* c)\mu$ is now easily seen to be a simple isomorphism.

Throughout the rest of this paper when no confusion will arise, we write π_1 instead of $\pi_1(W)$ to simplify notation.

Proof of Theorem 3.1. By Lemmas 3.3 and 3.4, it suffices to analyze $C_*(\hat{E}, \hat{E}_M)$ where $\hat{E}$ is the covering space of E corresponding to the kernel of $p_* : \pi_1(E) \to \pi_1(W)$. This covering space, however, is easy to identify: it is just the pull back via p of the universal cover $q_W : \tilde{W} \to W$ of W. Therefore there is a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & \hat{E} \arrow[r, "p'"] \arrow[d, "q"] & \tilde{W} \arrow[d, "q_W"] \\ F \arrow[r] & E \arrow[r, "p"] & W \end{tikzcd}$$

and $\hat{E}$ is a bundle over $\tilde{W}$ with fiber F as well as a principal π_1 bundle over E.

On the other hand since $(W^{n+1}; M^n, N^n)$ is an h-cobordism with $n \geq 5$, there is a handlebody presentation for W of the form $M \times I \cup \varphi_1{}^t \cup \cdots \cup \varphi_s{}^t \cup \varphi_1^{t+1} \cup \cdots \cup \varphi_s^{t+1}$ with handles only of index t and $t + 1$ for some $t \geq 3$. (In the case that M is a manifold with boundary, W is a relative h-cobordism and we assume that the handles do not meet ∂M.) Let $V = M \times I \cup \varphi_1{}^t \cup \cdots \cup \varphi_s{}^t$, $E_V = p^{-1}(V)$, and $\hat{E}_V = q^{-1}(E_V)$. Then there is an exact sequence

$$0 \to C_*(\hat{E}_V, \hat{E}_{M \times I}) \otimes R \to C_*(\hat{E}, \hat{E}_{M \times I}) \otimes R \to C_*(\hat{E}, \hat{E}_V) \otimes R \to 0$$

of free chain complexes over $R(\pi_1)$. It is obvious that we may choose $R(\pi_1)$ bases for these chain complexes satisfying the usual compatibility condition (cf. [4; p. 365]). Furthermore in proposition 4.2, we shall show that $H_*(\hat{E}, \hat{E}_V; R)$ and $H_*(\hat{E}_V, \hat{E}_{M \times I}; R)$ are $R(\pi_1)$ free and have natural bases. Since $H_*(\hat{E}, \hat{E}_{M \times I}) = 0$, the torsions of all these chain complexes are defined and [4; Theorem 3.2]

$$\Delta_R(E, E_{M \times I}) = \Delta_R(E, E_V) + \Delta_R(E_V, E_{M \times I}) + \Delta_R(\mathscr{H})$$

where $\mathscr{H}$ is the homology exact sequence

$$\cdots \to H_{n+1}(\hat{E}, \hat{E}_V; R) \to H_n(\hat{E}_V, \hat{E}_{M \times I}; R) \to H_n(\hat{E}, \hat{E}_{M \times I}; R) \to H_n(\hat{E}, \hat{E}_V; R) \to \cdots$$

Since $\Delta_R(E, E_{M \times I})$ obviously equals $\Delta_R(E, E_M)$, the proof is completed by evaluating each of the torsions on the right hand side of the above equation. These evaluations are carried out in the next two sections.

§4. THE EVALUATION OF $\Delta_R(\mathscr{H})$

It is the object of this section to evaluate $\Delta_R(\mathscr{H})$. In order to do this we must first show that $\mathscr{H}$ is free over $R(\pi_1)$ and specify preferred bases.

Toward this end let $\bar{C}_*$ be the chain complex with $\bar{C}_{t+1} = H_{t+1}(\tilde{W}, \tilde{V})$, $\bar{C}_t = H_t(\tilde{V}, \widetilde{M \times I})$, $\bar{C}_i = 0$ for $i \neq t + 1$, t and $\bar{\partial} : \bar{C}_{t+1} \to \bar{C}_t$ the boundary operator of the triple $(\tilde{W}, \tilde{V}, \widetilde{M \times I})$. Since $V = M \times I \cup \varphi_1{}^t \cup \cdots \cup \varphi_s{}^t$ and each $\varphi_i{}^t$ is a handle of index t, it is well known that $H_t(\tilde{V}, \widetilde{M \times I})$ is free over $Z(\pi_1)$ with one generator for each t-handle [4; p. 390]. These generators can be described geometrically as follows: let $f_i : (B_i{}^t \times B^{n+1-t}, S^{t-1} \times B^{n+1-t}) \to (V, M \times 1)$ be a PL embedding with image the handle φ_i, $i = 1, \ldots, s$. Let $\tilde{f}_i : B_i{}^t \times B^{n+1-t} \to \tilde{V}$ be a lift of f_i. Then $\{\tilde{f}_{i*}[B_i{}^t \times 0, S_i^{t-1} \times 0]\} \subset H_t(\tilde{V}, \widetilde{M \times I})$ $i = 1, \ldots, s$ form a basis $e^t = (e_1{}^t, \ldots, e_s{}^t)$, for $H_t(\tilde{V}, \widetilde{M \times I})$ over $Z(\pi_1)$.

Similarly $H_{t+1}(\tilde{W}, \tilde{V})$ is free over $Z(\pi_1)$ with a preferred basis $e^{t+1} = (e_1^{t+1}, \ldots, e_s^{t+1})$ where e_i^{t+1} is represented by the zero section of a $(t+1)$ handle lying over φ_i^{t+1}. It is well known that $\bar{C}_*$ is acyclic and that its torsion is just $\tau(W, M)$ (cf. [4; Theorem 9.3, p. 391]).

LEMMA 4.1. *There is a commutative diagram*

$$\begin{array}{ccc} \bar{C}_{t+1} \otimes H_m(F; R) & \xrightarrow{\psi_*} & H_{t+m+1}(\hat{E}, \hat{E}_V; R) \\ \downarrow{\scriptstyle \bar{\partial} \otimes 1} & & \downarrow{\scriptstyle \partial} \\ \bar{C}_t \otimes H_m(F; R) & \xrightarrow{\theta_*} & H_{t+m}(\hat{E}_V, \hat{E}_{M \times I}; R) \end{array}$$

with ψ_* and θ_* isomorphisms of R modules where ∂ is the boundary operator of the triple $(\hat{E}, \hat{E}_V, \hat{E}_{M \times I})$.

Proof. This follows immediately from the identification of the E^1 terms of the Serre spectral sequence given in Spanier [5; pp. 473–480]. (Although Spanier assumes the base space is a CW complex, this is used only in applying [5; Theorem 3, p. 474] and it is clear that this theorem holds for manifold pairs (X, A) where $A \subset \partial X$ with the filtration $X_\lambda = A \times I \cup$ (all handles of index $\leq \lambda$).)

Note that the conclusion of 4.1 is only that ψ_* and θ_* are isomorphisms of R modules. We wish to show that they are in fact isomorphisms of right $R(\pi_1)$ modules. We define therefore a right action of π_1 on $\bar{C}_i \otimes H_m(F; R)$ $i = t, t+1$ by setting $(c \otimes x)\alpha = c\alpha \otimes \sigma(\alpha)_* x$ where σ denotes the action of π_1 on $H_m(F; R)$ and extend this action linearly over $R(\pi_1)$, that is $(c \otimes x)\lambda = c\lambda \otimes \left(\sum_\alpha r_\alpha \sigma(\alpha)_* x\right)$ for $\lambda = \sum r_\alpha \alpha \in R(\pi_1)$.

LEMMA 4.2. *Let $\bar{C}_i \otimes H_m(F; R)$ be a right module over $R(\pi_1)$ as above. Then ψ_* and θ_* are isomorphisms of right $R(\pi_1)$ modules.*

Proof. The proof is sketched in an appendix, but the lemma essentially follows from the discussion in [5; pp. 473–480].

COROLLARY 4.3. *If $H_*(F; R)$ is free over R, then $H_*(\hat{E}, \hat{E}_V; R)$ and $H_*(\hat{E}_V, \hat{E}_{M \times I}; R)$ are free over $R(\pi_1)$.*

Proof. This is an immediate consequence of 4.2 and the proof of 1.3.

Now let $\mathscr{H} = (H_n, \partial_n)$ be the (acyclic) chain complex with $H_{3i} = H_i(\hat{E}, \hat{E}_V; R), H_{3i+1} = H_i(\hat{E}, \hat{E}_W; R)$, $H_{3i+2} = H_i(\hat{E}_V, \hat{E}_W; R)$, and boundary operators coming from the exact sequence of the triple $(\hat{E}, \hat{E}_V, \hat{E}_W)$. When $H_*(F; R)$ is free over R, 4.3 shows that these homology groups are free over $R(\pi_1)$ and may be endowed with preferred bases over $R(\pi_1)$ as follows: let $f^i = (f_1{}^i, \ldots, f^i_{m(i)}) \in H_i(F; R)$ be a basis for $H_i(F; R)$. Then $e^{t+1} \otimes f^i = (e_j^{t+1} \otimes f_k{}^i)$ and $e^t \otimes f^i = (e_j{}^t \otimes f_k{}^i)$ where $j = 1, \ldots, s$; $k = 1, \ldots, m(i)$ are $R(\pi_1)$ bases for $\bar{C}_{t+1} \otimes H_i(F; R)$ and $\bar{C}_t \otimes H_i(F; R)$ respectively. Let h_{3i} (respectively h_{3i-1}) be the basis of $H_{3i} = H_i(\hat{E}, \hat{E}_V; R)$ (respectively of $H_{3i-1} = H_{i-1}(\hat{E}_V, \hat{E}_W; R)$ corresponding to $e^{t+1} \otimes f^{i-(t+1)}$ under ψ_* (respectively to $e^t \otimes f^{i-(t+1)}$ under θ_*). Finally since $H_{3i+1} = H_i(\hat{E}, \hat{E}_W; R) = 0$, we take h_{3i+1} to be empty.

LEMMA 4.4. *Relative to the bases described above, if the action of π_1 on $H_*(F; R)$ is simple, then*

$$\Delta_R(\mathscr{H}) = \sum (-1)^i \sigma_{i*} \Delta_R(W, M).$$

Proof. We show first that for any acyclic based chain complex C_* with $C_{3i+1} = 0$ for all i, the torsion $\Delta(C_*)$ is given by $\sum (-1)^{3i}[c_{3i-1}/c_{3i}]$ where c_j is the distinguished bases for C_j and we are following Milnor's notation [4; p. 363]. For by definition [4; p. 365], $\Delta(C_*) = \sum (-1)^j [b_j b_{j-1}/c_j]$ since C_* is acyclic. Since $C_{3i+1} = 0$ for every i, $B_j = 0$ if $j = 3i$ or $3i+1$, while $B_j = C_j$ if $j = 3i-1$. Thus we may take b_j to be empty if $j = 3i$ or $3i+1$, and $b_j = c_j$ if $j = 3i-1$. Substituting these expressions for b_j in the general formula, and noting that $[c_{3i-1}/c_{3i-1}] = 0$, yields the result.

In particular $\Delta_R(\mathscr{H}) = \sum (-1)^{3i}[h_{3i-1}/h_{3i}]$. The matrix interpretation of the endomorphisms σ_{i*} given in Section 1, however, shows immediately that $[h_{3i-1}/h_{3i}] = \sigma_{i*}\Delta_R(W, M)$. Hence $\Delta_R(\mathscr{H}) = \sum (-1)^i \sigma_{i*}\Delta_R(W, M)$.

§5. THE EVALUATION OF $\Delta_R(E, E_V) + \Delta_R(E_V, E_{M\times I})$.

It is the object of this section to prove

LEMMA 5.1. *Relative to the bases for $C_*(\hat{E}, \hat{E}_V) \otimes R$, $C_*(\hat{E}_V, \hat{E}_{M\times I}) \otimes R$, $H_*(\hat{E}, \hat{E}_V; R)$, and $H_*(\hat{E}_V, \hat{E}_{M\times I}; R)$ over $R(\pi_1)$ described in Section 4*

$$\Delta_R(E, E_V) + \Delta_R(E_V, E_{M\times I}) = 0.$$

The proof requires a rather technical lemma which may be of some independent interest.

LEMMA 5.2 (Excision). *Let $f: (X, A) \to (Y, B)$ be a PL relative homeomorphism where (X, A) and (Y, B) are pairs in $\mathscr{P}$. Let $S \subset \pi_1(Y)$ be a normal subgroup and $G = \pi_1(Y)/S$. Let $q_Y : \hat{Y} \to Y$ be the regular covering space of Y corresponding to S and set $\hat{B} = q_Y^{-1}(B)$. Let $q : \hat{X} \to X$ be the principal G bundle over X induced by f and let $\hat{f} : \hat{X} \to \hat{Y}$ cover f. Let $\hat{A} = q^{-1}(A)$. Suppose $H_*(\hat{X}, \hat{A}; R)$ and $H_*(\hat{Y}, \hat{B}; R)$ are free $R(G)$ modules with preferred bases. If $\hat{f}_* : H_i(\hat{X}, \hat{A}; R) \to H_i(\hat{Y}, \hat{B}; R)$ is a simple isomorphism for all i, then $\Delta_R(X, A; G) = \Delta_R(Y, B; G)$.*

We recall that an isomorphism of based modules is simple if its matrix with respect to the given bases represents the zero element in the Whitehead group.

Note that if the map $f: (X, A) \to (Y, B)$ is an inclusion, then it is an excision since $f: X - A \to Y - B$ is a homeomorphism.

Proof of 5.2. Let (K, K_0) and (L, L_0) be triangulations of (X, A) and (Y, B) respectively such that $f: K \to L$ is simplicial. Let $(\hat{K}, \hat{K}_0)$ and $(\hat{L}, \hat{L}_0)$ be the induced triangulations of $(\hat{X}, \hat{A})$ and $(\hat{Y}, \hat{B})$ respectively and $C_*(\hat{K}, \hat{K}_0)$ and $C_*(\hat{L}, \hat{L}_0)$ be the cellular chain complexes of $(\hat{K}, \hat{K}_0)$ and $(\hat{L}, \hat{L}_0)$ regarded as free modules over $Z(G)$ equipped with natural bases. Since $f: (K, K_0) \to (L, L_0)$ is simplicial, so is $\hat{f}: (\hat{K}, \hat{K}_0) \to (\hat{L}, \hat{L}_0)$ and there is an induced map $\hat{f}_* : C_*(\hat{K}, \hat{K}_0) \to C_*(\hat{L}, \hat{L}_0)$ of $Z(G)$ modules.

Now let C_* be the "mapping cone" of $\hat{f}_*$; that is, the chain complex with $C_n = C_n(\hat{L}, \hat{L}_0) \otimes C_{n-1}(\hat{K}, \hat{K}_0)$ and $\partial(x, y) = (\partial x \otimes (-1)^{n-1} f(y), \partial y)$. If C_* is given the obvious bases over $Z(G)$, we have an exact sequence of based chain complexes $0 \to C_*(\hat{L}, \hat{L}_0) \xrightarrow{i} C_* \xrightarrow{j} C_*(\hat{K}, \hat{K}_0) \to 0$ where j has degree -1. Since the corresponding homology sequence $\mathscr{H}$ is just

$$\cdots \longrightarrow H_n(\hat{K}, \hat{K}_0) \xrightarrow{\hat{f}_*} H_n(\hat{L}, \hat{L}_0) \xrightarrow{i_*} H_n(C_*) \xrightarrow{j_*} H_{n-1}(\hat{K}, \hat{K}_0) \longrightarrow \cdots$$

and $\hat{f}_*$ is assumed to be an isomorphism, C_* is acyclic. Thus the torsions of all the chain complexes are defined, and $\Delta(C_*) = \Delta_R(Y, B; G) - \Delta_R(X, A; G) + \Delta(\mathscr{H}) = \Delta_R(Y, B; G) - \Delta_R(X, A; G)$ since $\hat{f}_*$ is a simple isomorphism. (The minus sign preceding $\Delta_R(X, A; G)$ comes from the fact that j has degree -1.) The proof is completed by showing that $\Delta(C_*) = 0$.

To see that $\Delta(C_*) = 0$, let M_f and $M_{f|}$ be the mapping cylinders of f and $f| : A \to B$. Let $\hat{M}_f$ be the covering space of M_f corresponding to S under the natural isomorphisms of $\pi_1(M_f)$ with $\pi_1(Y)$, and $\hat{M}_{f|}$ be the part of M_f over $M_{f|}$. Note that the part of $\hat{M}_f$ over $X \subset M_f$ is just $\hat{X}$. Then C_* is just the cellular chains on $(\hat{M}_f, \hat{M}_{f|} \cup \hat{X})$ relative to cells of the form $\sigma \times I$ and τ for $\sigma \in K - K_0$, $\tau \in L - L_0$. Thus $\Delta(C_*) = \Delta_R(M_f, M_{f|} \cup X; G)$. Since $f: K - K_0 \to L - L_0$ is a simplicial isomorphism, it is easy to see that M_f contracts to $M_{f|} \cup X$ and, therefore, that $\Delta_R(M_f, M_{f|} \cup X; G) = 0$ completing the proof.

Now let A be a free $R(\pi_1)$ module with generators $a_1, \ldots, a_s$ and $C_*(F)$ be the cellular chains on F relative to some fixed cellular structure $\{\tau_j | j \in J\}$. Then $C_*(F) \otimes R \otimes A$ is a free $R(\pi_1)$ module with basis $\tau_j \otimes 1 \otimes a_i$ $j \in J$, $i = 1, \ldots, s$, where the last tensor product is taken over R.

LEMMA 5.3. *Let R be a principal ideal domain. Then*

(i) *$H_*(C_*(F) \otimes R \otimes A)$ is free over $R(\pi_1)$ and has a natural $R(\pi_1)$ basis. Hence the torsion $\Delta(C_*(F) \otimes R \otimes A)$ is defined.*

(ii) $\Delta_R(E_V, E_{M \times I}) = (-1)^t \Delta(C_*(F) \otimes R \otimes A)$.

(iii) $\Delta_R(E, E_V) = (-1)^{t+1} \Delta(C_*(F) \otimes R \otimes A)$.

Lemma 5.1 follows trivially from 5.3.

Proof. Since $C_*(F) \otimes R$ is a free chain complex over the principal ideal domain R, the Universal Coefficient Theorem yields the short exact sequence

$$0 \longrightarrow H_i(F; R) \otimes A \xrightarrow{\mu} H_i(C_*(F) \otimes R \otimes A) \longrightarrow \mathrm{Tor}[H_{i-1}(F; R), A] \longrightarrow 0.$$

Since A is free over R, the Tor term vanishes and μ is an isomorphism. Since $H_i(F, R)$ is free over R with generators $f_1{}^i, \ldots, f^i_{m(i)}$, and A is free over $R(\pi_1)$ on generators $a_1, \ldots, a_s$, $H_i(F; R) \otimes A$ becomes a free $R(\pi_1)$ module on the generators $f_k{}^i \otimes a_l$ $k = 1, \ldots, m(i)$, $l = 1, \ldots, s$ where $R(\pi_1)$ operates only on the second factor. Thus (i) holds.

The proof of (ii) involves an application of the Excision Lemma. Let $f_i : (B_i{}^t \times B^{n+1-t}, S_i^{t-1} \times B^{n+1-t}) \to (V, M \times I)$ $i = 1, \ldots, s$ be the PL embeddings of Section 4 whose images are the t-handles of V not in $M \times I$. Then $\bigcup_{i-1}^s f_i$ is a relative PL homeomorphism. Since $B_i{}^t \times B^{n+1-t}$ is contractible, $p : E \to W$ pulls back via f_i to a trivial bundle, and f_i can be covered by a bundle map $g_i : (B_i{}^t \times B^{n+1-t} \times F, S_i^{t-1} \times B^{n+1-t} \times F) \to (E_V, E_{M \times I})$. Then $\bigcup_{i=1}^s g_i$ is a relative PL homeomorphism. We note that g_i is not unique.

Consider the π_1 bundle over $B_i{}^t \times B^{n+1-t} \times F$ induced from $q : \hat{E} \to E$ by g_i. Since $q : \hat{E} \to E$ is induced from the universal cover of W, it is easy to see that this bundle is trivial. Hence g_i can be covered by a π_1 equivariant map $\hat{g}_i : (B_i{}^t \times B^{n+1-t} \times F \times \pi_1, S_i^{t-1} \times B^{n+1-t} \times F \times \pi_1) \to (\hat{E}_V, \hat{E}_{M \times I})$ where the action of π_1 on $B_i{}^t \times B^{n+1-t} \times F \times \pi_1$ is just right multiplication in the last factor. Since $\bigcup_{i=1}^s \hat{g}_i$ is a relative

PL homeomorphism, $\bigcup_{i=1}^{s}\hat{g}_{i*}: H_*(\bigcup(B_i^t, S_i^{t-1}) \times B^{n+1-t} \times F \times \pi_1) \to H_*(\hat{E}_V, \hat{E}_{M\times I})$ is an isomorphism of modules over $Z(\pi_1)$.

In order to show $\bigcup_{i=1}^{s}\hat{g}_{i*}$ is a simple isomorphism, g_i and $\hat{g}_i$ must be chosen with some care. This is done as follows: let $\tilde{f}_i : (B_i^t \times B^{n+1-t}, S_i^{t-1} \times B^{n+1-t}) \to (\tilde{V}, \widetilde{M \times I})$ be the lift of f_i given in Section 4. Then $\tilde{f}_{i*}[B_i^t \times 0, S_i^{t-1} \times 0] = e_i^t \in H_t(\tilde{V}, \widetilde{M \times I})$. Let $\overline{g}_i : B_i^t \times B^{n+1-t} \times F \to \hat{E}_V$ be an admissible lift (cf. the appendix for the definition) of $\tilde{f}_i$, and extend $\overline{g}_i$ to $\hat{g}_i : B_i^t \times B^{n+1-t} \times F \times \pi_1 \to \hat{E}_V$ by using the action of π_1. Finally let $g_i = q\overline{g}_i$. Then $\hat{g}_i$ covers g_i since $q\hat{g}_i(x, y, z, \alpha) = q[\hat{g}_i(x, y, z, e)\alpha] = q\hat{g}_i(x, y, z, e) = q\bar{g}_i(x, y, z) = g_i(x, y, z)$.

Now let $H_{j+t}(\bigcup(B_i^t, S_i^{t-1}) \times B^{n+1-t} \times F \times \pi_1 : R) = H_t(\bigcup(B_i^t, S_i^{t-1}) \times \pi_1) \otimes H_j(F; R)$ have the $R(\pi_1)$ basis $[B_i^t \times e, S_i^{t-1} \times e] \otimes f_k^j$ $i = 1, \ldots, s$, $k = 1, \ldots, m(j)$ where $(f_1^j, \ldots, f_{m(j)}^j)$ is the R basis of $H_j(F; R)$. The definition of the isomorphism θ_* of 4.1 and the fact that $\bar{g}_i$ is an admissible lift of $\tilde{f}_i$ show immediately that $\bigcup_{i=1}^{s}\hat{g}_{i*}$ is a simple isomorphism. The Excision Lemma now applies and

$$\Delta_R(E_V, E_{M\times I}) = \Delta_R \bigcup (B_i^t \times B^{n+1-t} \times F, \bigcup S_i^{t-1} \times B^{n+1-t} \times F).$$

Since the inclusion

$$j : (\bigcup B_i^t \times 0 \times F, \bigcup S_i^{t-1} \times 0 \times F) \subset (\bigcup B_i^t \times B^{n+1-t} \times F, \bigcup S_i^{t-1} \times B^{n+1-t} \times F)$$

is a simple equivalence and induces a simple isomorphism of homology over $R(\pi_1)$ relative to the obvious $R(\pi_1)$ bases, $\Delta_R(E_V, E_{M\times I}) = \Delta_R(\bigcup B_i^t \times F, \bigcup S_i^{t-1} \times F)$. By the combinatorial invariance of torsions $\Delta_R(\bigcup B_i^t \times F, \bigcup S_i^{t-1} \times F) = \Delta_R(C_* \otimes R)$ where C_* denotes the cellular chains on $(\bigcup B_i^t \times F \times \pi_1, \bigcup S_i^{t-1} \times F \times \pi_1)$ relative to any cellular decomposition.

In particular let B_i^t be given a cell structure with only one t-cell σ_i^t, F be given the cellular structure $\{\tau_j | j \in J\}$ described above, and π_1 have one zero cell α for each $\alpha \in \pi_1$. Then C_* is the free abelian group on the cells $\sigma_i^t \times \tau_j \times \alpha$ $i = 1, \ldots, s$; $j \in J$; $\alpha \in \pi_1$; and the boundary operator is just $\partial(\sigma_i^t \times \tau_j \times \alpha) = (-1)^t \sigma_i^t \times \partial\tau_j \times \alpha$.

Define $h_k : C_k(F) \otimes R \otimes A \to C_{k+t} \otimes R$ by setting

$$h_k(\tau_j \otimes 1 \otimes a_i \alpha) = (-1)^{kt}(\sigma_i^t \times \tau_j \times \alpha) \otimes 1$$

and extending linearly over R. (This makes sense since $\tau_j \times 1 \times a_i \alpha$ where $j \subset J$; $i = 1, \ldots, s$; $\alpha \in \pi_1$; is an R basis for $C_*(F) \otimes R \otimes A$.) Since π_1 acts on C_* by right multiplication of the last factor h_k is an $R(\pi_1)$ module homomorphism. In fact h_k is a simple isomorphism. It is easy to check that $h = \{h_k\}$ is a chain map of degree t and that $h_* : H_*(C_*(F) \otimes R \otimes A) \to H_*(C_* \otimes R)$ is a simple isomorphism. A simple calculation using Milnor's definition of torsions, now shows that $\Delta_R(C_* \otimes R) = (-1)^t \Delta_R(C_*(F) \otimes R \otimes A)$ completing the proof of (ii).

The proof of (iii) is similar to that of (ii) and 5.3 is established.

§6. APPENDIX

This appendix contains a sketch of the proof of 4.2. We follow the notation and terminology of [5; pp. 473–480] with one exception: we will call a bundle map $\bar{\sigma} : \Delta^k \times F \to E$ covering a map $\sigma : \Delta^k \to B$ an *admissible lift* if for some vertex $v \in \Delta^k$ there is a path $\rho : I \to B$

with $\rho(0) = b_0$ and $\rho(1) = \sigma(v)$ and a bundle map $A : I \times F \to E$ covering ρ such that $A|0 \times F$ is the inclusion $F = p^{-1}(b_0) \subset E$ and $A|1 \times F$ is isotopic to $\bar{\sigma}|v \times F$. The existence of admissible lifts follows as in [5; p. 477] by using the Bundle Covering Homotopy Theorem (cf. [6; p. 50]).

Now let $\sigma_i : (\Delta^{t+1}, \dot{\Delta}^{t+1}) \to (\tilde{W}, \tilde{V})$ be a singular simplex representing the generator $e_i^{t+1} \in H_{t+1}(\tilde{W}, \tilde{V})$ and let $\bar{\sigma}_i : (\Delta^{t+1}, \dot{\Delta}^{t+1}) \times F \to (\hat{E}, \hat{E}_V)$ be an admissible lift of σ_i.

LEMMA. *Let $\alpha \in \pi_1$ and let h be any homeomorphism representing $\sigma(\alpha^{-1})$. Then the composite $\Delta^{t+1} \times F \xrightarrow{1 \times h} \Delta^{t+1} \times F \xrightarrow{\bar{\sigma}_i} \hat{E} \xrightarrow{\alpha} \hat{E}$ is an admissible lift of $\alpha\sigma_i$.*

Proof. Since $\overline{\sigma_i}$ is an admissible lift of σ_i, there is a vertex $v \in \Delta^{t+1}$, a path $\rho : I \to \tilde{W}$ with $\rho(0) = \tilde{w}_0$ and $\rho(1) = v$, and a bundle map $A : I \times F \to \hat{E}$ covering ρ such that $A|0 \times F$ is the inclusion and $A|1 \times F$ is isotopic to $\overline{\sigma_i}|v \times F$. By altering A using the isotopy we assume that $A|1 \times F = \overline{\sigma_i}|v \times F$ and form $\overline{\sigma_i} \vee A$ with domain $(\Delta^{t+1} \vee I) \times F$ where $1 \in I$ is identified with $v \in \Delta^{t+1}$. To prove the lemma it suffices by [5; lemma 11, p. 477] to show that for some point $x \in \Delta^{t+1} \vee I, \alpha(\overline{\sigma_i} \vee A)(1 \times h)|x \times F$ is an admissible lift of $\alpha(\sigma_i \vee \rho)|x$. We show this for $x = \tilde{w}_0\alpha$, where α is written on the right to emphasize that we think of covering transformations as acting on the right.

Let $\tau : I \to \tilde{W}$ be a path such that $\tau(0) = \tilde{w}_0$ and $\tau(1) = \tilde{w}_0\alpha$ and let $T : I \times F \to \hat{E}$ be an admissible lift of τ. Since $\hat{E} = \{(e, y) \in E \times \tilde{W} | p(e) = q_W(y)\}$, T may be considered to be a pair of functions (T_1, T_2) satisfying $pT_1 = q_W T_2$. Since $T_2 = p'T = \tau$, the condition $pT_1 = q_W T_2 = q_W \tau$ means that T_1 is a lift of $q_W\tau$ with $T_1|0 \times F$ the inclusion $F \subset E$. Since $q\tau$ is a loop in W, $T_1|1 \times F$ represents $\sigma(q\tau)$. The identification of the group of covering transformations with π_1, however, shows that $q\tau$ represents α^{-1}. Hence $T_1|1 \times F$ is isotopic to h and τ and T may be altered so that $T_1|1 \times F = h$. Then $T_1(1, y) = (h(y), \tilde{w}_0\alpha) = (h(y), \tilde{w}_0)\alpha = [(\overline{\sigma_i} \vee A)(1 \times h)(1, y)]\alpha = \alpha(\overline{\sigma_i} \vee A)(1 \times h)(1, y)$ where we have abused our convention of writing covering transformations on the right. The lemma follows.

The proof of 4.2 is now easy; for by definition $\psi_*((\sigma_i\alpha) \otimes \sigma(\alpha)_* x) = \overline{\alpha_1\sigma}(\xi \otimes \sigma(\alpha)_* x) = [\alpha\overline{\sigma_1}(1 \times h)]_*(\xi \otimes \sigma(\alpha)_* x) = (\alpha\overline{\sigma_i})_*(\xi \otimes x) = \alpha_*\psi_*(\xi \otimes x) = [\psi_*(\xi \otimes x)]\alpha$ for any $\alpha \in \pi_1$, where the bar denotes an admissible lift and $\xi \in H_{t+1}(\Delta^{t+1}, \dot{\Delta}^{t+1})$ is a generator.

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